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EXCITATION OF THE TRAVELING WAVE TUBE

by

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Abstract

In a traveling wave guide, there exists one mode which propagates with a velocity less than the velocity of light. This report contains an investigation of the excitation of this mode by means of a coaxial line. It is found that the strength of the helical field components depend on the solution of a set of linear equations. The expansion of an arbitrary function in terms of the eigen-functions associated with the helix fields is dealt with.

1. Introduction

The present report is a continuation of the work done by this group* in the analysis of the helical type traveling wave tube. It treats the problem of determining the strength of the modes excited in the cold helix in terms of the amplitude of an arbitrary incident field. In particular the case when the lowest mode of a coaxial line is fed in is discussed.

The important mode of the helix is the one that travels slower than the velocity of light. Conditions are derived which indicate how the strength of this mode may be increased. As an illustration of the method, a practical case was calculated in detail. It was found that the conditions were violated and that the important mode was not strongly excited.

To carry out the analysis, it was found necessary to show that the radial components of the symmetrical modes of the helix form a complete orthnormal set of eigen-functions. This is done in the Appendix by a method similar to that of E. C. Titchmarsh: "Eigen Function Expansions".

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1. R. S. Phillips, A Helical Wave Guide. New York University, Washington Square College. Research Report No. 170-2.
2. R. S. Phillips and Henry Malin, A Helical Wave Guide II. New York University, Washington Square College. Research Report No. 170-3.
3. R. S. Phillips and Henry Malin, Investigation of Exceptional Modes: $n = \pm 1, \pm 2$. New York University, Washington Square College. Research Report No. 170-7.
4. Bernard Friedman and Henry Malin, Propagation in Wave Guides Bounded by Anisotropic Plates. New York University, Washington Square College. Research Report No. 170-8.
5. Bernard Friedman, Amplification of Traveling Wave Tube. New York University, Washington Square College. Research Report No. TW-9.
6. W. Sollfrey, Propagation along a Helical Wire. New York University, Washington Square College. Research Report No. TW-10.
7. R. S. Phillips, The Electromagnetic Field Produced by a Helix. New York University, Washington Square College. Research Report No. TW-11.

2. Formulation of the Problem

The model which we use is the familiar idealization of the helix as a cylindrical sheath of radius b which is perfectly conducting in a helical direction and perfectly non-conducting in the direction perpendicular to this. It is found that there are in general an infinite number of non-attenuated modes of propagation along such a guide: some of these modes have phase velocities along the guide greater than the velocity of light in free space and others have smaller phase velocities. R. S. Phillips*, to whom this work is due, shows how, by the proper choice of design parameters of the guide, it is possible to eliminate all modes having phase velocities along the guide greater than the free space velocity of light. We shall be concerned only with the lowest modes set up [those with no angular dependence] and furthermore we need those modes whose phase velocity is greater than the free space velocity of light. The inclusion of the latter modes is necessary for a representation theory.

For a monochromatic source the electromagnetic field independent of the angle θ inside an infinitely long circular cylinder can be represented in the form:

$$
\begin{aligned}
E_r &= A\frac{ih}{\lambda} J_1(\lambda r) F_o , & H_r &= B\frac{ih}{\lambda} J_1(\lambda r) F_o \\
E_\theta &= B\frac{i\mu\omega}{\lambda} J_1(\lambda r) F_o , & H_\theta &= A\frac{ik^2}{\mu\omega\lambda} J_1(\lambda r) F_o \\
E_z &= A\, J_o(\lambda r) F_o , & H_z &= B\, J_o(\lambda r) F_o
\end{aligned}
\tag{2.1}
$$

In these relations

$$\lambda^2 = k^2 - h^2 , \quad F_o = e^{ihz-i\omega t} \tag{2.2}$$

*Op. Cit. (1)

If the boundary conditions—vanishing of E and H in the helical direction—implicit in the idealization are applied, there results

$$(2.3)\qquad \begin{aligned} &\alpha J_0(\lambda b)\,A - \frac{i\mu\omega b}{\lambda} J_1(\lambda b)\,B = 0 \\ &\frac{ik^2 b}{\mu\omega\lambda} J_1(\lambda b)\,A + \alpha J_0(\lambda b)\,B = 0 \end{aligned}$$

In order that there exists a non-trivial solution for A and B, the determinant of the coefficients must vanish; that is

$$(2.4)\qquad + \frac{\tan\psi}{bk} = \frac{J_1(\lambda b)}{\lambda b\, J_0(\lambda b)}\,, \qquad (2.4)' \quad - \frac{\tan\psi}{bk} = \frac{J_1(\lambda' b)}{\lambda b\, J_0(\lambda' b)}$$

For instance E_r may be exhibited in the form

$$E_r = i\sum\left(\frac{a_j h_j}{\lambda_j} J_1(\lambda_j r) + \frac{a_j' h_j'}{\lambda_j} J_1(\lambda_j' r)\right)$$

where the summation is over the roots of Eqs. (2.4), (2.4')

Eq. (2.4) has an infinity of real roots and 2 pure imaginary roots. There are no roots for λ complex. Furthermore if the roots of Eq. (2.4) with the + sign are denoted by $\lambda_1, \lambda_2, \lambda_3, \ldots$ those with the — sign by $\lambda_1', \lambda_2', \lambda_3', \ldots$, then it can be shown that any function $f(r)$ such that $\int r^{1/2} f(r)\,dr$ converges can be expanded in the series.

$$(2.5)\qquad \sum_n a_n J_1(\lambda_n r) + a_n' J_1(\lambda_n' r), \quad \left[\text{where } \lambda_n, \lambda_n' \text{ are the roots of Eq. (2.4)}\right]$$

This question will be treated more completely in Appendix I.

The relevance of the foregoing paragraph to the problem on hand is as follows: The field components for the co-ax are known expansions in terms of Bessel functions of argument r. Hence these components are expressible as series of the form Eq. (2.5). But (see Eq. (2.1)) the E_r, E_θ, H_r and H_θ components of the helix are precisely, aside from the factor F_0 series of the form Eq. (2.5). Hence the equality of E_r, E_θ, H_r and H_θ for the helix to the corresponding terms of the co-ax expresses on the one hand the continuity of the

tangential components and on the other the expansion of the co-ax-components into a series of the form Eq. (2.5)

For the coaxial guide the field components are:*

E-wave $H_r = H_z = E_\phi = 0$

(2.6)
$$E_r = \left\{ \frac{h^*}{\epsilon\omega} \left(AJ_1(r\gamma) + B\,Y_1(r\gamma) + \frac{c_1}{r} \right\} F^* $$
$$E_z = \frac{i}{\epsilon\omega}\gamma \left\{ AJ_0(r\gamma) + B\,Y_0(r\gamma) \right\} F^*$$
$$H_\phi = \left\{ AJ_1(r\gamma) + B\,Y_1(r\gamma) + \frac{c}{r} \right\} F^*$$

where

(2.7) $$F^* = e^{ih^*z - i\omega t}\ , \quad \gamma^2 = k^2 - h^{*2}$$

and

(2.8) $$-\frac{A}{B} = \frac{Y_0(a\gamma)}{J_0(a\gamma)} = \frac{Y_0(b\gamma)}{J_0(b\gamma)}$$

H-wave $E_r = E_z = H_\phi = 0$

(2.6')
$$E_\phi = \left\{ C\,J_1(r\gamma') + D\,Y_1(r\gamma') \right\} F^{**}$$
$$H_r = -\frac{h^{**}}{\omega\mu} E_\phi\ F^{**}$$
$$H_z = -\frac{i\gamma'}{\omega\mu} \left\{ C\,J_0(r\gamma') + D\,Y_0(r\gamma') \right\} F^{**}$$

(2.7') $$F^{**} = e^{ih^{**}z - i\omega t}\ , \quad \gamma'^2 = k^2 - h^{**2}$$

and

(2.8') $$-\frac{C}{D} = \frac{Y_1(a\gamma')}{J_1(a\gamma')} = \frac{Y_1(b\gamma')}{J_1(b\gamma')}\ .$$

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*Sarbacher and Edson: Hyper and Ultra High Frequency Engineering, page 276.

The component E_r , for example, is seen to be

$$E_r = \sum \frac{h_n^*}{\epsilon\omega} (A_n J_1(r \gamma_n) + B_n Y_1(r \gamma_n)) F_n$$

where the summation is taken over the roots of Eq. (2.18). The remaining components can be written as similar sums.

3. Matching of the Components.

The boundary conditions of the problem are clearly satisfied by the continuity of the tangential components of the fields of the coaxial guide and helix at the interface of the two structures. Thus we want

$$(E_r)_{co-ax} = (E_r)_{helix} \;;\; (E_\theta)_{co-ax} = (E_\theta)_{helix}; \; (H_r)_{co-ax} = (H_r)_{helix}; (H_\theta)_{co-ax} = (H_r)_{heli}$$

which become in virtue of the expressions for the fields given in the preceding section

$$(3.1) \quad \frac{c_1}{r} e^{-i\omega t - ih_o^* \ell} + \frac{1}{\epsilon\omega} \sum \left\{ A_n J_1(r \gamma_n) + B_n Y_1(r \gamma_n) \right\} h_n^* e^{-i\omega t - ih_n^* \ell}$$

$$= i \sum \left\{ \frac{a_n h_j}{\lambda_j} J_1(\lambda_j r) + \frac{a_j' h_j'}{\lambda_j'} J_1(\lambda_j' r) \right\} e^{-i\omega t}$$

$$(3.2) \sum \left\{ C_n J_1(r \gamma_n') + D_n Y_1(r \gamma_n') \right\} e^{-i\omega t - ih_n^{**}\ell}$$

$$= -i\mu\omega \sum \left\{ \frac{b_j}{\lambda_j} J_1(\lambda_j r) + \frac{b_j'}{\lambda_j'} J_1(\lambda_j' r) \right\} e^{-i\omega t}$$

$$(3.3) \quad -\frac{1}{\omega\mu} \sum h_n^{**} (C_n J_1(r \gamma_n') + D_n Y_1(r \gamma_n') e^{-i\omega t - ih_n^{**}\ell}$$

$$= i \sum \left\{ \frac{b_j h_j}{\lambda_j} J_1(\lambda_j r) + \frac{b_j' h_j'}{\lambda_j'} J_1(\lambda_j' r) \right\} e^{-i\omega t}$$

$$(3.4) \quad \frac{c}{r} e^{-i\omega t - ih_o^* \ell} + \sum (A_n J_n(r \gamma_n) + B_n Y_1(r \gamma_n)) e^{-i\omega t - ih_n^* \ell}$$

$$= \frac{ik^2}{\mu\omega} \sum \left\{ \frac{a_j}{\lambda_j} J_1 (\lambda_j - r) + \frac{a_j'}{\lambda_j} J_1 (\lambda_j' r) \right\} e^{-i\omega t}$$

The summations are taken over the roots (infinite in number) of Eqs. (2.4), (2.4)', (2.8), (2.8'), and ℓ is the length of the co-axial line.

The problem of finding the field components of the helix is reduced to finding the coefficients a_j, a'_j, b_j and b'_j. However, it is known that the eigenfunctions $\left\{1,\ A_n J_1(r\gamma_n) + B_n Y_1(r\gamma_n)\right\}$, which furnish the field components of the coaxial guide from an orthogonal set. Hence, if Eqs. (3.1) - (3.4) are multiplied by

$1,\quad A_n J_1\left(r\gamma_n + B_n Y_1(r\gamma_n)\right)$; $\qquad C_n\left(J_1(r\gamma'_n) + D_n Y_1(r\gamma'_n)\right)$;

$C_n\left(J_1(r\gamma'_n) + D_n Y_1(r\gamma'_n)\right)$; $1,\quad A_n J_1\left(r\gamma_n + B_n Y_1(r\gamma_n)\right)$, respectively, and integrated with respect to r from r = a to r = b, there results the six sets of equations:

$$(3.5)\quad e^{-h_0^*\ell} c_1 \log \frac{b}{a} = -i\sum \frac{a_j h_j}{\lambda_j^2}\left[J_o(\lambda_j b) - J_o(\lambda_j a)\right]$$

$$-i\sum \frac{a'_j h'_j}{\lambda'_j}\left[J_o(\lambda'_j b) - J_o(\lambda'_j a)\right],$$

$$(3.6)\quad \frac{B_n h_n^*}{\pi\gamma_n\varepsilon\omega}\left[\frac{1}{J_o^2(b\gamma_n)} - \frac{1}{J_o^2(a\gamma_n)}\right] = -i\sum \frac{a_j h_j}{\lambda_j^2 - \gamma_n^2}\left[\frac{J_o(\lambda_j a)}{J_o(\gamma_n a)} - \frac{J_o(\lambda_j b)}{J_o(\gamma_n b)}\right]$$

$$-i\sum \frac{a'_j h'_j}{\lambda'^2_j - \gamma_n^2}\left[\frac{J_o(\lambda'_j a)}{J_o(\gamma_n a)} - \frac{J_o(\lambda'_j b)}{J_o(\gamma_n b)}\right],$$

$$(3.7)\quad \frac{D_n}{\pi\gamma'^2_n}\left[\frac{1}{J_1^2(b\gamma'_n)} - \frac{1}{J_1^2(a\gamma'_n)}\right] = i\mu\omega\sum \frac{b_j}{\lambda_j(\lambda_j^2 - \gamma'^2_n)}\left[\frac{J_1(\lambda_j b)}{J_1(\gamma'_n b)} - \frac{J_1(\lambda_j a)}{J_1(\gamma'_n a)}\right]$$

$$+ i\mu\omega\sum \frac{b'_j}{\lambda'_j(\lambda'^2_j - \gamma'^2_n)}\left[\frac{J_1(\lambda'_j b)}{J_1(\gamma'_n b)} - \frac{J_1(\lambda'_j a)}{J_1(\gamma'_n a)}\right],$$

$$(3.8)\quad \frac{D_n h_n^{**}}{\pi\gamma_n'^2}\left[\frac{1}{J_1^2(b\gamma_n)} - \frac{1}{J_1^2(a\gamma_n)}\right] = i\mu\omega\sum \frac{b_j h_j}{\lambda_j^2(\lambda_j^2-\gamma_n'^2)}\left[\frac{J_1(\lambda_j b)}{J_1(\gamma_n' b)} - \frac{J_1(\lambda_j a)}{J_1(\gamma_n' a)}\right]$$

$$+ i\mu\omega\sum \frac{b_j h_j}{\lambda_j(\lambda_j^2-\gamma_n'^2)}\left[\frac{J_1(\lambda_j' b)}{J_1(\gamma_n' b)} - \frac{J_1(\lambda_j a)}{J_1(\gamma_n' a)}\right].$$

$$(3.9)\quad c\log\frac{b}{a}\, e^{-h_o \ell} = -\frac{ik}{\mu\omega}\sum\frac{a_j}{\lambda_j^2}\left[J_o(\lambda_j b) - J_o(\lambda_j a)\right] - \frac{ik}{\mu\omega}\sum\frac{a_j'}{\lambda_j'^2}\left[J_o(\lambda_j' b) - J_o(\lambda_j' a)\right]$$

and

$$(3.10)\quad \frac{B_n}{\pi\gamma_n}\left[\frac{1}{J_o^2(b\gamma_n)} - \frac{1}{J_o^2(a\gamma_n)}\right] = \frac{ik}{\mu\omega}\sum\frac{a_j}{\lambda_j^2-\gamma_n^2}\left[\frac{J_o(\lambda_j b)}{J_o(\gamma_n b)} - \frac{J_o(\lambda_j a)}{J_o(\gamma_n a)}\right]$$

$$+ \frac{ik}{\mu\omega}\sum\frac{a_j'}{\lambda_j^2-\gamma_n^2}\left[\frac{J_o(\lambda_j' b)}{J_o(\gamma_n b)} - \frac{J_o(\lambda_j' a)}{J_o(\gamma_n a)}\right].$$

If the D_n and B_n are eliminated in the above equations, we get

$$(3.11)\quad i\sqrt{\frac{\mu}{\varepsilon}}\, c\, e^{-h_o^* \ell}\log\frac{b}{a} = \sum\frac{a_j h_j}{\lambda_j^2} L_j + \sum\frac{a_j' h_j'}{\lambda_j'^2} L_j'$$

$$(3.12)\quad i\,\frac{c}{\omega\varepsilon}\, e^{-h_o^* \ell}\log\frac{b}{a} = \sum\frac{a_j}{\lambda_j^2} L_j + \sum\frac{a_j'}{\lambda_j'^2} L_j'$$

$$(3.13)\quad 0 = \sum\frac{a_j}{h_j + h_n^*} N_{j,n} + \sum\frac{a_j'}{h_j' + h_n^*} N_{j,n}'$$

$$(3.14)\quad 0 = -\sum\frac{a_j}{\lambda_j(h_j + h_n^{**})} Q_{j,n} + \sum\frac{a_j'}{\lambda_j'(h_j' + h_j^{**})} Q_{j,n}'$$

where the b_j, b_j' have been eliminated with the aid of Eqs. (2.3) and where we

have set

$$(3.15)\qquad L_j = J_o(\lambda_j b) - J_o(\lambda_j a) \quad , \quad L_j' = J_o(\lambda_j' b) - J_o(\lambda_j' a)$$

$$(3.16)\qquad N_{j,n} = \frac{J_o(\lambda_j b)}{J_o(\gamma_n b)} - \frac{J_o(\lambda_j a)}{J_o(\gamma_n a)} \; , \; N_{j,n}' = \frac{J_o(\lambda_j' b)}{J_o(\gamma_n b)} - \frac{J_o(\lambda_n' a)}{J_o(\gamma_n a)}$$

$$(3.17)\qquad Q_{j,n} = \frac{J_1(\lambda_j b)}{J_1(\gamma_n' b)} - \frac{J_1(\lambda_j a)}{J_1(\gamma_n' a)} \; , \; Q_{j,n} = \frac{J_1(\lambda_j' b)}{J_1(\gamma_n' b)} - \frac{J_1(\lambda_j' a)}{J_1(\gamma_n' a)} \; .$$

The solution of the problem rests with the determination of the unknown coefficients a_j, a_j' by means of the linear equations $\big((3.11) - (3.14)\big)$. Eqs. (311) and (3.12) are non-homogeneous and the non-zero left hand terms arise from the principal mode of the coaxial guide. The infinite set of homogeneous equations $\big((3.13$ and $(3.14)\big)$ arise from the cut-off modes of the co-ax. It is hoped that the higher cut-off modes of the co-ax do not seriously affect the field components of the helix.

A numerical example will now be given for $k = .628$, $b = .36$ cm., $a = .15$ cm., $\Theta = 5^o$, we get

λ_j	h_j	L_j	N_j	Q_j
$\lambda_1 = 8.44$	$h_1 = 8.42i$	$L_1 = -.91$	$N_1 = -2.49$	$Q_1 = -1.87$
$\lambda_2 = 16.44$	$h_2 = 16.43i$	$L_2 = .16$	$N_2 = -.57$	$Q_2 = -.08$
$\lambda_1' = 4.44i$	$h_1' = 2.20$	$L_1' = .64$	$N_1' = -36.61$	$Q_1' = -3.77i$
$\lambda_2' = 13.94$	$h_2' = 13.93i$	$L_j' = -.23$	$N_2' = 1.92$	$Q_2' = -.03$

The unknowns a_1, a_2, a_1', a_2' are then found to be

$$a_1 = \sqrt{\frac{\mu}{\varepsilon}}\;\frac{I}{2\pi}\;e^{-h_o^*\ell}\;\log\frac{b}{a}\left[14.34 - 105.95i\right]$$

$$a_2 = \sqrt{\frac{\mu}{\varepsilon}}\;\frac{I}{2\pi}\;e^{-h_o^*\ell}\;\log\frac{b}{a}\left[478.27 - 2677.49i\right]$$

$$a_1' = \sqrt{\frac{\mu}{\varepsilon}}\ \frac{I}{2\pi}\ e^{-h_0^* \ell}\ \log \frac{b}{a}\ \left[2.50 - 24.84i\right]$$

$$a_2' = \sqrt{\frac{\mu}{\varepsilon}}\ \frac{I}{2\pi}\ e^{-h_0^* \ell}\ \log \frac{b}{a}\ \left[234.55 - 872.34i\right]$$

where I is the longitudinal current carried by the inner conductor of the co-ax and $h_0^* = ik$. If these values are substituted in the relevant equations $\big((3.1) - (3.4)\big)$ we have the components E_r, E_θ, H_r and H_θ. E_z and H_z are found from expansions of the form

$$E_z = \sum a_n J_1(\lambda_n r)\, F_0 \quad , \qquad H_z = \sum b_n J_1(\lambda_n r)\, F_0 \, .$$

The coefficient associated with the slow mode is a_1'. It is seen from the above example that a_1' is smaller than the coefficients a_1, a_2, a_2'. Hence the dominant term in the field components will not be that of the slow mode. An examination of the Eqs. $\big((3.11) - (3.14)\big)$ indicates that the design parameters should be so chosen that the coefficients in these equations should be small compared to L_1', N_1', Q_1'.

APPENDIX

On the Completeness of the Set of Functions $\left\{ r^{1/2} J_1 (\lambda_n r) \right\}$ -

1. Introduction

In an investigation of an idealized helical wave guide of radius and pitch angle Θ R.S. Phillips* was led to the transcendental equations

$$(1.1) \qquad \frac{\tan \Theta}{ak} = - \frac{n^2}{x^2} \sqrt{1 + \frac{x^2}{\alpha^2}} \pm \frac{1}{x} \frac{J_n'(x)}{J_n(x)}, \quad n = 0,1,2,\ldots,$$

where α is a constant dependent on the pitch angle. The roots of Eq. (1.1) are related to the modes (both attenuated and non-attenuated) set up by the helical guide. He showed that for n a fixed integer (i) Eq. (1.1) has an infinitude of roots for x real (ii) Eq. (1.1) has a finite number of roots for x pure imaginary. Thus the question of complex roots is still open. We shall show in this note that Eq. (1.1) for the special but important case $n = 0$ has no complex roots.

For $n = 0$, Eq. (1.1) reduces to

$$(1.2) \qquad \frac{\tan \Theta}{ak} = \pm \frac{1}{x} \frac{J_0'(x)}{J_0(x)} .$$

It will also be shown that the set of functions $\left\{ r^{1/2} J_1 (\lambda_n r) \right\}$ where the λ_n are the infinitude of roots of Eq. (1.2), is closed. The main theorem upon which our work rests is given in E. C. Titchmarsh: Eigenfunction Expansions Associated with Second Order Differential Equations.

2. We are going to investigate the set of functions satisfying the differential equation:

$$(2.1) \qquad \frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \left(\mu - \frac{1}{r^2} \right) u = 0$$

subject to the homogeneous boundary conditions

$$(2.2) \qquad u(o) = 0, \quad h \left[u'(r) - \frac{1}{r} u(r) \right] + u(r) = 0, \text{ at } r = a,$$

where h and a are arbitrary constants.

*R. S. Phillips, A Helical Wave Guide. New York University, Washington Square College. Research Report No. 170-2.

Let

$$(2.3) \qquad u(r) = r^{-1/2}\, v(r)$$

The differential equation (2.1) becomes

$$(2.4) \qquad \frac{d^2v}{dr^2} + \left(\mu - \frac{1 - 1/4}{r^2}\right) v = 0$$

and Eqs. (2.2) take the form

$$(2.5) \qquad v(o) = 0\,, \quad h\left[\frac{d}{dr}(r^{-1/2}v) - r^{-3/2}\, v)\right] + r^{-1/2}\, v = 0,\ r = a.$$

Let

$$(2.6) \qquad v_1 = r^{1/2}\, J_1\,(\sqrt{\mu}\, r)$$

$$(2.7) \qquad v_2 = A\, r^{1/2}\, J_1\,(\sqrt{\mu}\, r) + B\, r^{1/2}\, Y_1\,(\sqrt{\mu}\, r)$$

v_1 and v_2 are solutions of Eq. (2.4). Apply the first boundary condition to Eq. (2.6). It is satisfied since $J_1(o) = 0$. The second boundary condition applied to Eq. (2.7) yields

$$(2.8) \quad h\left[A\sqrt{\mu}\; J_1'\,(\sqrt{\mu}\, a) + B\sqrt{\mu}\; Y_1'\,(\sqrt{\mu}\, a) - \frac{1}{a}\, A\, J_1(\sqrt{\mu}\, a) - \frac{1}{a}\, B\, Y_1\,(\sqrt{\mu}\, a)\right]$$
$$+ A\, J_1(\sqrt{\mu}\, a) + B\; Y_1\,(\sqrt{\mu}\, a) = 0,$$

or

$$(2.9) \quad A\left[h\sqrt{\mu}\; J'\,(\sqrt{\mu}\, a) + J_1(\sqrt{\mu}\, a) - \frac{h}{a}\; J_1\,(\lambda\, a)\right]$$

$$= -\,B\left[h\sqrt{\mu}\, Y_1'\,(\sqrt{\mu}\, a) + Y_1\,(\sqrt{\mu}\, a) - \frac{h}{a}\; Y_1\,(\sqrt{\mu}\, a)\right]$$

Set

$$(2.10) \qquad A = -\left[h\sqrt{\mu}\; Y_1'\,(\sqrt{\mu}\, a) + \frac{a-h}{a}\; Y_1\,(\sqrt{\mu}\, a)\right]$$

$$(2.11) \qquad B = \left[h\sqrt{\mu}\; J_1'\,(\sqrt{\mu}\, a) + \frac{a-h}{a}\; J_1\,(\sqrt{\mu}\, a)\right]$$

$$(2.12) \qquad \phi\,(r,\sqrt{\mu}) = r^{1/2}\, J_1\,(\sqrt{\mu}\, r)$$

$$(2.13) \qquad \psi\,(r,\sqrt{\mu}) = A\; r^{1/2}\; J_1\,(\sqrt{\mu}\, r) + B\, r^{1/2}\, Y_1\,(\sqrt{\mu}\, r)$$

The Wronskian of ϕ and ψ is readily computed, it is

$$(2.14) \qquad W(\phi\ ,\ \psi\ ) \ = \ \frac{2}{\pi}\ B \ = \omega(\sqrt{\mu}\ ) \quad \text{say.}$$

Finally let

$$v = \frac{\psi(r,\sqrt{\mu})}{\omega(\sqrt{\mu}\ )}\int_0^r \phi(r',\sqrt{\mu})\ f(r')\ dr' + \frac{\phi(r,\sqrt{\mu})}{\omega(\lambda)}\int_r^a \psi(r',\sqrt{\mu})f(r')\ dr' .$$

It will now be shown that v is the solution of the differential equation

$$(2.16) \qquad \frac{d^2v'}{dr^2} + \left(\mu - \frac{3/4}{r^2}\right)\ v(r) \ = r(r)$$

which satisfies the boundary conditions (2.5).

We have

$$(2.17) \qquad v' \ = \ \frac{\psi'(r,\sqrt{\mu})}{\omega(\sqrt{\mu})}\int_0^r \phi(r',\lambda)\ f(r')\ dr' + \frac{\phi'(r,\sqrt{\mu})}{\omega(\sqrt{\mu})}\int_r^a \psi(r',\sqrt{\mu})f(r')$$

$$(2.18) \qquad v'' \ = \ \frac{\psi''(r,\sqrt{\mu})}{\omega(\sqrt{\mu})}\int_0^r \phi(r',\lambda)f(r')\ dr' + \frac{\phi''(r,\sqrt{\mu})}{\omega(\sqrt{\mu})}\int_r^a \psi(r',\sqrt{\mu})f(r')\ d$$

$$+ \frac{W(\phi\ ,\ \sqrt{\mu}\ )}{\omega(\sqrt{\mu}\ )}\ f(r) \ .$$

Thus

$$(2.19) \qquad v''(r) + \left(\mu - \frac{3/4}{r^2}\right)\ v(r) - f(r) \equiv$$

$$\left[\psi''(r,\sqrt{\mu}) + \left(\mu - \frac{3/4}{r^2}\right)\psi(r)\right]\ \frac{1}{\omega(\sqrt{\mu})}\int_0^r \varphi(r',\sqrt{\mu})\ f(r')\ dr'$$

$$\left[\varphi''(r,\sqrt{\mu}\ ) = \left(\mu - \frac{3/4}{r^2}\right)\ \varphi(r,\ \sqrt{\mu}\ \right]\ \frac{1}{\omega(\sqrt{\mu}\ )}\int_r^a \psi(r',\sqrt{\mu}\ )f(r')\ dr'$$

$$+ f(r) - f(r) \equiv 0 \ .$$

Furtheromore,

$$(2.20) \qquad v(0) = 0$$

and

$$(2.21) \quad h\left[\frac{d}{dr}\left(r^{-1/2}\ v(r)\right) - r^{-3/2}\ v(r)\right] + r^{-1/2}\ v(r)$$

$$\equiv A\left[h\sqrt{\mu}\,J_1'\ (\sqrt{\mu}\ r) - \frac{h}{a}\ J_1\ (\sqrt{\mu}\ r) + J_1\ (\sqrt{\mu}\ r)\right]$$

$$+ B\left[h\sqrt{\mu}\,Y_1'\ (\sqrt{\mu}\ r) - \frac{h}{a}\ Y_1(\sqrt{\mu}\ r) + Y_1\ (\sqrt{\mu}\ r)\right] \qquad \text{at} \quad r = a.$$

The right hand side of Eq. (2.21) is zero in virtue of the definitions of A and B [Eqs. (2.10) and (2.11)].

To summarize, then, we have shown that the function v defined by Eq. (2.15) satisfies the differential equation (2.16) with the boundary conditions (2.5).

3. Now let μ_n be a root of Eq. (2.14), ie.,

(3.1) $$\omega(\sqrt{\mu_n}) = 0, \quad \text{or} \quad a\sqrt{\mu_n} \; \frac{J_1'(\sqrt{\mu_n}a)}{J_1(\sqrt{\mu_n}a)} = \frac{h-a}{h} \; .$$

Thus

(3.2) $$W(\phi, \psi) = 0.$$

Hence, by a well known property of Wronskians

(3.3) $$\psi(r, \sqrt{\mu_n}) = k_n \; \phi(r, \sqrt{\mu_n})$$

where

$$k_n = A(\sqrt{\mu_n}) = -\left[h\sqrt{\mu_n}\, Y_1'(\sqrt{\mu_n}a) + \frac{a-h}{a} Y_1(\sqrt{\mu_n}\, a)\right]$$

$$= - \frac{-h\pi}{2a\, J_1(\sqrt{\mu_n}a)}$$

The functions $r^{1/2} J_1(\sqrt{\mu_n}\, r)$, $r^{1/2} J_1(\sqrt{\mu_m} r)$ are orthogonal in the interval (0,a). μ_n and μ_m represent any two distinct roots of Eq. (3.1). We have [See Watson: Bessel Functions p.135]

(3.5) $$\int_0^a r\; J_1(\sqrt{\mu_n}r)\, J_1(\sqrt{\mu_m}r)\, dr$$

$$= \frac{\sqrt{\mu_n}\, J_1(\sqrt{\mu_m}a)\; J_1'(\sqrt{\mu_n}a) - \sqrt{\mu_m}\; J_1(\sqrt{\mu_n}a)\; J_1'(\sqrt{\mu_m}a)}{\sqrt{\mu_n} - \sqrt{\mu_m}}$$

$$= \frac{J_1(\sqrt{\mu_n}a)\, J_1(\sqrt{\mu_m}a)}{a(\sqrt{\mu_n} - \sqrt{\mu_m})}\left[\sqrt{\mu_n}\, a\, \frac{J_1'(\sqrt{\mu_n}a)}{J_1(\sqrt{\mu_n}a)} - \sqrt{\mu_m}\, a\, \frac{J_1'(\sqrt{\mu_m}a)}{J_1(\sqrt{\mu_m}a)}\right]$$

$$= \frac{J_1(\sqrt{\mu_n}a)\, J_1(\sqrt{\mu_m}a)}{a(\sqrt{\mu_n} - \sqrt{\mu_m})}\left[\frac{h-a}{h} - \frac{h-a}{h}\right] = 0 \; .$$

Also

$$(3.6)\quad \int_0^a r\, J_1^2(\sqrt{\mu_n} r)\, dr = \frac{1}{2}\, a^2 \left\{ \left(1 - \frac{1}{\mu_n a^2}\right) J_1^2(\sqrt{\mu_n} r) + J_1'^2(\sqrt{\mu_n} a) \right\}$$

A calculation shows that the right hand side of Eq. (3.6) is given by $k_n / \omega'(\sqrt{\mu_n})$.

Hence we have the result that the set of functions

$$(3.7)\quad \frac{r^{1/2}\, J_1(\sqrt{\mu_n} r)}{\sqrt{k_n / \omega(\sqrt{\mu})}}$$

is an orthonormal set.

The set of functions $\dfrac{r^{1/2} J_1(\sqrt{\mu_n} r)}{\sqrt{k_n / \omega'(\sqrt{\mu_n})}}$ is orthonormal.

The next question which arises is the representation of an arbitrary function f(r) as an infinite series in terms of the above set. Titchmarsh* has shown that any function f(r) such that

(3.8) $\int_0^a r^{1/2} f(r)\, dr$ is absolutely convergent can be expanded in terms of the Bessel functions $r^{1/2} J_1(\sqrt{\mu_n} r)$, where the μ_n are roots of the Eq. (3.1), according to the Fourier representation

$$(3.8)\quad f(r) = \sum \frac{k_n}{\omega'(\sqrt{\mu_n})}\, r^{1/2}\, J_1(\sqrt{\mu_n}\, r) \int_0^a r'^{1/2}\, J_1(\sqrt{\mu_n} r')\, f(r')\, dr' .$$

Furthermore, Eq. (3.1) has an infinity of real roots, zero or two pure imaginary roots and no complex roots.

Similar theorems hold if the left hand side of Eq.. (3.1) is replaced by its negative.

Finally we wish to show the relevance of Eq. (3.1) to Eq. (1.2). In Eq. (3.1)

$$(3.1)\quad \omega(\lambda) = h\sqrt{\mu_n} a\, J_1'(\sqrt{\mu_n} a) + (a-h)\, J_1(\sqrt{\mu_n}\, a) = 0$$

replace $\sqrt{\mu_n}\, a\, J_1'$ by

$$(3.9)\quad \sqrt{\mu_n}\, a\, J_1'(\sqrt{\mu_n} a) = \sqrt{\mu_n} a\, J_0(\sqrt{\mu_n} a) - J_1(\sqrt{\mu_n} a) .$$

* E. C. Titchmarsh: Eigenfunction Expansions Associated with Second Order Differential Equations. Ch. I.

There results

$$h \sqrt{\mu_n} a \left\{ J_o \left(\sqrt{\mu_n} a\right) - \frac{1}{\sqrt{\mu_n}\, a} J_1 \left(\sqrt{\mu_n} a\right) \right\} + (a-h) J_1 \left(\sqrt{\mu_n} a\right) = 0 \tag{3.10}$$

which reduces to

$$\text{const.} = \frac{J_o'(\sqrt{\mu_n} a)}{\sqrt{\mu_n}\, a\, J_o(\sqrt{\mu_n} a)} \tag{3.11}$$

$$= \frac{J_o'(x)}{x\, J_o(x)} \; .$$

This is Eq. (1.2) and our proof is complete.

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